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D-Brane Stability and Monodromy

Paul S. Aspinwall

Center for Geometry and Theoretical Physics, Duke University
Box 90318, Durham, NC 27708-0318
E-mail: psa@cgtp.duke.edu

Michael R. Douglas

Department of Physics and Astronomy, Rutgers University
Piscataway, NJ 08855-0849, and
Institut des Hautes Études Scientifiques
Bures-sur-Yvette, 91440 France
E-mail: mrd@physics.rutgers.edu

Abstract: We review the idea of Π-stability for B-type D-branes on a Calabi-Yau manifold. It is shown that the octahedral axiom from the theory of derived categories is an essential ingredient in the study of stability. Various examples in the context of the quintic Calabi-Yau threefold are studied and we plot the lines of marginal stability in several cases. We derive the conjecture of Kontsevich, Horja and Morrison for the derived category version of monodromy around a "conifold" point. Finally, we propose an application of these ideas to the study of supersymmetry breaking.

Keywords: D-branes, Differential and Algebraic Geometry.

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1. Introduction

How exactly are we to consider a D-brane on a Calabi-Yau space X? The basic picture would be some cycle $C \subset X$ perhaps equipped with some kind of vector bundle $E \to C$. There has to be more to D-branes than this however. It has been known since the early days of mirror symmetry, from works such as Candelas et al [1], that the general notion of a cycle $C \subset X$ is far from clear and perhaps not even well-defined.

Suppose one begins with an even-dimensional cycle lying in some homology class. The simplest example would be a point whose homology class lies in H_0. Upon "parallel transport" around a loop in the moduli space of the complexified Kähler form, $B+iJ$, this homology class is generally transformed into some other element of H^{even}. Thus the point appears to have transformed magically into a general collection of even-dimensional cycles.

Based in the ideas of Mukai [2, 3], Kontsevich [4] realized that this monodromy appeared fairly naturally in the more abstract setting of the derived category of coherent sheaves. A more direct analysis of D-branes in terms of the derived category was then pioneered in [5] together with further analysis in [6, 7, 8].

One of the purposes of this paper is to show explicitly what happens physically to the D-branes as we undergo such monodromy. We focus purely on even-dimensional (B-type) D-branes. In order to do this we have to first study the notion of stability of a D-brane which is itself a very interesting subject.

The notion of stability we use is the Π-stability of [9]. As generalized in [5], this is a criterion determining which elements in the derived category actually correspond to physical BPS D-branes, on any Calabi-Yau with any moduli. The generality of this proposal comes with a price, and many of the issues which come up in practice remain to be resolved. We discuss some of these, and illustrate them with several examples of D-branes on the quintic Calabi-Yau threefold. Some of the points we raise were also discussed in [10]. Other recent work on D-branes on Calabi-Yau's includes [11]–[15].

It is perhaps worth emphasizing here that the subjects of the derived category and Π-stability are probably central to an understanding of many aspects of string theory. While we focus here purely on D-branes, it was observed in [16] that the derived category, and presumably Π-stability, are fairly unavoidable in the heterotic string. (See [17, 18], for example, for some comments on stability in the classical sense.) Indeed this kind of approach may well be required for the complete study of any Yang–Mills (or at least hermitean–Yang–Mills) theory in a stringy context.

In section 2 we review the basic ideas of the origin of the derived category in the context of D-branes and the appearance of Π-stability. In order to obtained a well-defined notion of stability we show that the "octahedral axiom" of triangulated categories in required.

In section 3 we clarify the analysis of section 2 by applying it to the quintic Calabi-Yau threefold. This section contains several plots showing the lines of marginal stability of several D-branes.

In section 4 we show how monodromy manifests itself physically as a result of stability considerations.

Finally, in section 5 we discuss an application to the study of supersymmetry breaking.

2. Π-stability

In this section we review and elaborate on the notions of Π-stability for objects in the derived category of coherent sheaves. The course of logical steps required to understand Π-stability is somewhat complicated so we will try to spell out each step of the argument.

2.1 Topological D-branes

Let us begin with the theory of topological B-branes as discussed in [5, 6]. The definition of the topological B model is at first sight independent of Kähler moduli, and at first sight topological B branes will be just the boundary conditions discussed by Witten [19], associated with holomorphic cycles carrying holomorphic bundles.

However, this is not all of the topological boundary conditions. This point can be motivated in many ways. For example, the typical Calabi-Yau stringy Kähler moduli space as determined by mirror symmetry has several large volume limits producing spaces of distinct topology; this leads to several distinct sets of holomorphic bundles, while the complete set of boundary conditions must be the same in all such limits.

There is a natural mathematical construction which, starting from any of these sets (actually categories) of boundary conditions, leads to a universal result: to pass from the original category to the derived category. Implicit in Kontsevich's homological mirror symmetry proposal [4] is the claim that topological branes correspond precisely to objects in the derived category. The more recent proposals of [5, 6] explain physically how to construct such boundary conditions, and why this generalization is necessary.

Initially the boundaries in Witten's B-model are taken to be a set of finite-dimensional vector bundles. We then further distinguish boundary conditions by associating a number $n \in \mathbb{Z}$ known as the "grade", to each possible boundary. Let us denote a bundle associated with a boundary labeled by n as E^n.

A topological open string stretching from E^m to F^n then lives in the Hilbert subspace $H^p(X, (E^m)^\vee \otimes F^n)$, where p is the intrinsic "bulk" ghost number of the open string. At this point we have used the one-to-one relation between states in topological string theory, Ramond ground states in the corresponding physical string (and thus massless fermions in space-time), and to the (Dolbeault) cohomology of the $\bar{\partial}$ operator coupled to the bundle $(E^m)^\vee \otimes F^n$.

The total ghost number in the topological field theory is then redefined as follows: an open string between branes with abstract grades m and n, i.e. in $H^p(X, (E^m)^\vee \otimes F^n)$, is *defined* to have ghost number $p - m + n$.

The wedge product gives us an associative product on cohomology:

$$H^p\left(X, (E^m)^\vee \otimes F^n\right) \times H^q\left(X, (F^n)^\vee \otimes G^p\right) \to H^{p+q}\left(X, (E^m)^\vee \otimes G^p\right). \tag{2.1}$$

In topological field theory terms this is the operator product. At this point we switch to the language of algebraic geometry.[1] We replace the notion of holomorphic vector bundle with the associated coherent sheaf. These sheaves, to be thought of as D-branes, together with their morphisms, to be thought of as open strings, form the structure of a category.

This category is in fact an abelian category: this means that every map between sheaves has a kernel and cokernel. Gaining this property is one reason to generalize from bundles to sheaves; it is only true in this larger context. Physically, the first point at which sheaves enter is in considering the zero size instanton, as explained in [20]. Thus, the vector bundle E^n is generalized to a sheaf $\mathscr{E}^n$, and the linear vector space $H^p(X, (E^m)^\vee \otimes E^n)$ becomes the linear vector space $\mathrm{Ext}^p(\mathscr{E}^m, \mathscr{E}^n)$.

The next step is to allow objects E which are direct sums of objects E^n with different abstract grades n, and to allow a BRST operator which is a general map $Q : E \to E$ of grade 1 satisfying $Q^2 = 0$. In other words, each B-brane can be written as a complex of

[1] It would be interesting to know if any of these considerations can be generalized for differential geometry and less or no supersymmetry.

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finite length:

$$\cdots \longrightarrow \mathscr{E}^{-1} \xrightarrow{d_{-1}} \underline{\mathscr{E}^0} \xrightarrow{d_0} \mathscr{E}^1 \xrightarrow{d_1} \cdots, \tag{2.2}$$

where the boundary maps d^n represent the various components of the BRST operator Q. Such a generalization respects all of the axioms of topological field theory; it can be further argued that this generalization is in fact the general ghost number one deformation of the topological field theory [6].[2] Some notations from [6] are used as follows: an underline in a complex represents the "zeroth" position. Also, if $\mathscr{F}$ is a coherent sheaf then $\underline{\mathscr{F}}$ will represent a complex with the only nonzero entry being $\mathscr{F}$ at the zeroth position. We also use the conventional notation that $\mathscr{E}^\bullet[q]$ represents the complex $\mathscr{E}^\bullet$ shifted q places to the *left*.

We will denote the complex (2.2) as $\mathscr{E}^\bullet$. Note that the original notion of "grade" of a sheaf has now become "position" in a complex.

As argued in [5], many complexes correspond to the same topological brane, distinguished by adding brane-antibrane pairs or other combinations of branes which can annihilate to the vacuum. To factor out this redundancy, two complexes are taken to represent the same D-brane if and only if they are related by a *quasi-isomorphism.* This equivalence was explained in a slightly different way in [6]. The final result is that topological B-branes are (up to isomorphism) in one to one correspondence with objects in $\mathbf{D}(X)$ — the bounded derived category of coherent sheaves on X.

2.2 Cones and triangles

Further marginal deformations of the topological field theory of section 2.1 yields nothing new in the sense that the resulting D-branes remain objects in $\mathbf{D}(X)$. Given a pair of complexes $\mathscr{E}^\bullet$ and $\mathscr{F}^\bullet$ together with a chain map $f : \mathscr{E}^\bullet \to \mathscr{F}^\bullet$, the "mapping cone" $\mathrm{Cone}(f : \mathscr{E}^\bullet \to \mathscr{F}^\bullet)$ is defined as an object in $\mathbf{D}(X)$ given by

$$\longrightarrow \underline{\mathscr{E}^1 \oplus \mathscr{F}^0} \xrightarrow{\left(\begin{smallmatrix} d_E & 0 \\ f & d_F \end{smallmatrix}\right)} \mathscr{E}^2 \oplus \mathscr{F}^1 \xrightarrow{\left(\begin{smallmatrix} d_E & 0 \\ f & d_F \end{smallmatrix}\right)} \mathscr{E}^3 \oplus \mathscr{F}^2 \longrightarrow \cdots \tag{2.3}$$

It was argued in [6] that this is a general feature of topological D-branes. Namely two D-branes $\mathscr{E}^\bullet, \mathscr{F}^\bullet$ in $\mathbf{D}(X)$ are marginally deformed by a map f into $\mathrm{Cone}(f : \mathscr{E}^\bullet \to \mathscr{F}^\bullet)$. Thus the mapping cone represents the "sum" of two branes.

In the simplest case of the mapping cone we may think of $\mathscr{E}^\bullet$ as a the complex given by the single sheaf $\mathscr{E}$ at position zero and $\mathscr{F}^\bullet$ as a the complex given by the single sheaf $\mathscr{F}$ at position zero. A map $f : \mathscr{E} \to \mathscr{F}$ then yields the mapping cone as a complex:

$$0 \longrightarrow \mathscr{E} \xrightarrow{f} \underline{\mathscr{F}} \longrightarrow 0\ . \tag{2.4}$$

Reversing this construction one may view the complex

$$\cdots \longrightarrow \mathscr{E}^{-1} \xrightarrow{d_{-1}} \underline{\mathscr{E}^0} \xrightarrow{d_0} \mathscr{E}^1 \xrightarrow{d_1} \cdots \tag{2.5}$$

[2]See [7] for a further generalization which uses non-holomorphic forms, $\bar\partial\omega \neq 0$. At present there is no evidence that this further generalization is required to describe BPS states.

as an iteration of cones:

$$\mathrm{Cone}(\mathrm{Cone}(\mathrm{Cone}(\mathrm{Cone}(\cdots \to \mathscr{E}^{-1}) \to \mathscr{E}^{0}) \to \mathscr{E}^{1}) \to \cdots)\,. \tag{2.6}$$

In this way one may view the mapping cone as the basic operation in the construction of complexes.

This leads one to study the structure of the category of topological D-branes as a *triangulated category.* A triangle in $\mathbf{D}(X)$ is an sequence of morphisms of the form:

$$\begin{tikzcd} & C \arrow[dl, "{[1]}"', "w"] & \\ A \arrow[rr, "u"] & & B \arrow[ul, "v"'] \end{tikzcd} \tag{2.7}$$

The "[1]" denotes that the map w increases the grade of any object by one. That is, in terms of elements in the complexes, $w : c^n \to a^{n+1}$.

There must always be exactly one edge in a triangle with a "[1]". This edge may be moved however. For example, the triangle in (2.7) could be rewritten as

$$\begin{tikzcd} & C[-1] \arrow[dl] & \\ A \arrow[rr] & & B \arrow[ul, "{[1]}"'] \end{tikzcd} \tag{2.8}$$

The triangle (2.7) in $\mathbf{D}(X)$ is *distinguished* if and only if C is isomorphic to $\mathrm{Cone}(u : A \to B)$. We refer to chapter 5 of [21] for further details. Note that because the "[1]" can be shuffled around to any edge of the triangle, any vertex of a distinguished triangle can be considered the mapping cone on the opposing side of the triangle.

This democracy between the vertices of a distinguished triangle cannot be over-emphasized. Physically, it means that if two topological D-branes A and B can combine to give C, then A and $C[-1]$ can combine to give B etc. The exact meaning of this will be become more clear when we talk about physical D-branes below.

2.3 Physical branes

We would now like to relate the topological branes above to physical branes. We show in this section that many of the objects of $\mathbf{D}(X)$ can represent potential physical D-branes. In section 2.4 we will see that they need to satisfy further stability requirements in order to be physical.

Let us return to the state of affairs before we introduced the derived category and think of our D-branes as 6-branes wrapped on the Calabi-Yau. Such a brane is characterized by a choice of vector bundle E over X: the D6 charge is the rank of E, while the lower D-brane charges are determined by the Chern character of E. We will summarize this information in the RR charge vector $Q_i(E)$.

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This is a BPS state with a particular central change $Z(E)$. For an even-dimensional B-brane such as this, $Z(E)$ depends purely on the complexified Kähler form, $B+iJ$, and not on the complex structure of X. For large radius we have [22, 23]

$$Z=\sum_i Q_i(E)\Pi^i=\int_X e^{-B-iJ}\,ch(E)\sqrt{td(T_X)}+\text{quantum corrections} \tag{2.9}$$

We may compute $Z(E)$ exactly by passing to the mirror of X as we will see in section 3.2.

The basic physical interpretation of $Z(E)$ is that $M(E)=|Z(E)|$ is the mass of the BPS brane E,[3] while the phase $\arg Z(E)$ determines the particular $\mathscr{N}=1\subset\mathscr{N}=2$ supersymmetry which is preserved by the brane E. The phase of $Z(E)$ plays many roles in the subsequent considerations. One of the most familiar is in defining walls of marginal stability or "ms-walls."[4] Suppose we have two 6-branes E and F which might combine to give a bound state G. A well known argument using $Z(G)=Z(E)+Z(F)$ and the triangle inequality shows that either

- $\arg Z(E)=\arg Z(F)$ implying G is marginally stable for this decay into E and F, or
- $\arg Z(E)\neq\arg Z(F)$ and then, assuming G exists as a BPS state, G is stable against this decay.

Thus a given decay process can only take space on a specific (real) codimension one wall in moduli space, the ms-wall, determined by the charges $Q(E)$ and $Q(F)$. Of course being on the wall is a necessary but not sufficient condition for a decay.

We thus define the "grade" of the brane E to be

$$\varphi(E)=\frac{1}{\pi}\operatorname{Im}\log Z(E)\,. \tag{2.10}$$

While the specific $\mathscr{N}=1$ supersymmetry which is preserved by E is determined by $\varphi(E)$ up to shifts in $2\mathbb{Z}$, we wrote $\operatorname{Im}\log Z$ for the phase to emphasize that (as in [5]) we define the grade to take values in $\mathbb{R}$, not just $[0,2)$, by making a choice of branch for the logarithm.

To better understand this point, consider a pair of inequivalent D-branes whose values of φ happen to agree at a particular value of $B+iJ$. As we vary $B+iJ$, the difference between the grades of the branes varies continuously with $B+iJ$ but may exceed 2 even if we remain in a fundamental region of the moduli space. As we argue shortly, this relative grade has physical meaning.

We therefore require a more precise definition of φ as follows. Fix a basepoint in the moduli space $\mathscr{M}$ near large radius limit. For a given D-brane use (2.10) to fix φ at the basepoint in some fixed range, say, $0\leq\varphi<2$. Now demand that φ varies *continuously* along any path in the moduli space. This makes φ a well-defined function on the *path space* $\mathscr{P}$ consisting of paths in the moduli space starting at the fixed basepoint.

[3]This is true only if we use normalized central charges, i.e. defined as periods of a normalized three-form on the mirror satisfying $1=\int\Omega\wedge\bar{\Omega}$. We will not use the normalized masses in our subsequent considerations; clearly these are also valuable data (as seen in [11]) and it would be interesting to find some way to use them in our considerations.

[4]Of course if $\dim_{\mathbb{C}}\mathscr{M}=1$ these are lines of marginal stability, or ms-lines.

This definition fixes most of the ambiguity but not all. First, there is an arbitrary choice of the phase of the holomorphic three-form (and thus all the central charges $Z(E)$) at the basepoint. In fact, shifting φ for all branes simultaneously by a constant value will be physically meaningless; only differences $\varphi(E) - \varphi(F)$ will enter our considerations.

Second, one wants φ to be a well-defined function on *Teichmüller space* $\mathscr{T}$, which we define as a cover of moduli space on which central charges are single valued, or as the moduli space of mirror quintics carrying a basis of $H^3(X,\mathbb{Z})$. This cover should be branched only at singular points, with branchings determined by the monodromies on RR charges. Now all physical quantities should be well-defined functions on such a Teichmüller space, but in general $\varphi(E)$ as we have defined it may not be. If the central charge $Z(E)$ vanishes at a non-singular point P in moduli space (this is possible if the brane E does not physically exist at P), paths encircling P will shift $\varphi(E)$ by ± 2. We will discuss this point in section 2.4, eventually concluding that $\varphi(E)$ is well-defined only for stable objects, and that one should consider only paths which remain in the region of stability.

The fundamental physical reason we need to lift φ to $\mathbb{R}$ is that variations of the grade (2.10) determine the variation of the masses of open strings stretched between branes [5]. More precisely, the topological open strings discussed above correspond in the physical CFT to Ramond ground states, each of which will have a NS partner defined using spectral flow. The $\mathscr{N} = 2$ superconformal algebra determines the mass of the corresponding space-time boson in terms of its U(1) charge Q to be

$$m^2 = \frac{1}{2}(Q-1)\,. \tag{2.11}$$

We then combine this with the result that flow of grading affects the U(1) charge of a string between a brane E and a brane F as

$$\Delta Q = \Delta\varphi(F) - \Delta\varphi(E)\,. \tag{2.12}$$

In particular, if E and F are *neighbors* in the sense that their grades differ by one, then an element $f \in \mathrm{Hom}(E,F)$ corresponds to a marginal deformation to a new state, the mapping cone $G = \mathrm{Cone}(f : E \to F)$.

As we saw in section 2.2, we may use the mapping cone to iteratively build long complexes along the lines of (2.6). That is one follows the following algorithm. Let us assume that the complex begins with $\mathscr{E}^0$ as the leftmost term for simplicity.

1. Compute $\varphi(\mathscr{E}^0)$ and $\varphi(\mathscr{E}^1)$ near large radius limit using (2.9). Define them to lie in some fixed range, say $0 \le \varphi < 2$.
2. Let $A = \mathscr{E}^0$ and $B = \mathscr{E}^1$.
3. Analytically continue the φ's of A and B along a path in the moduli space $\mathscr{M}$ to find a point where $\varphi(B) = \varphi(A) + 1$.
4. This defines the new object $\mathrm{Cone}(A \to B)$ with $ch = ch(B) - ch(A)$ and $\varphi = \varphi(B)$.
5. Replace A by $\mathrm{Cone}(A \to B)$ and B by the next term in the complex.
6. Repeat from step 3 until the complex is exhausted.

Note that we will have more to say about step 3 at the end of section 2.4.

The property of the grading of the cone with respect to the constituent objects we used in the rules above is justified as follows. Referring to equation (2.7) if we have an open string u stretching from A to B and $\varphi(B) = \varphi(A) + 1$, then A and B form a marginally bound state $C = \mathrm{Cone}(u : A \to B)$. In this case $Z(C) = Z(A) + Z(B)$ where recall A was viewed as an *anti*-brane. Viewing A as a brane, we have $Z(C) = Z(B) - Z(A)$, which implies

$$\varphi(C) = \varphi(B) = \varphi(A) + 1\,. \tag{2.13}$$

This equation defines $\varphi(C)$ and, in particular, fixes the $2\mathbb{Z}$ ambiguity.

In this way we can hope to build a good many objects of $\mathbf{D}(X)$. An example of a complex which *cannot* be built is

$$0 \to \mathscr{F} \to 0 \to \mathscr{F} \to 0\,, \tag{2.14}$$

where $\mathscr{F}$ is any sheaf. One might claim the first three terms as giving $\mathrm{Cone}(\mathscr{F} \to 0)$ but then the relative φ of this cone and the second $\mathscr{F}$ is stuck at 2 for all values of $B + iJ$.

This particular object is decomposable (it is a direct sum of simpler objects) and as such would not correspond to a single physical brane; however it is likely that many indecomposable objects in $\mathbf{D}(X)$ also cannot be constructed using the above algorithm. Π-stability implies that such objects never correspond to physical branes.

This definition of the grade of a complex is somewhat complicated as it requires finding a succession of paths in moduli space, one for each step in the procedure. In special cases one might be able to simplify this. For example, one might try to build a complex

$$\cdots \longrightarrow \mathscr{E}^{-1} \xrightarrow{d_{-1}} \underline{\mathscr{E}^{0}} \xrightarrow{d_0} \mathscr{E}^{1} \xrightarrow{d_1} \cdots \tag{2.15}$$

in a single step, by finding a value for $B + iJ$ for which $\varphi(\mathscr{E}^n) = n + \alpha$ for all n for which $\mathscr{E}^n$ is nonzero, and some fixed real number α. However, such a value is not likely to exist if there are more than two nontrivial terms in the complex.

2.4 Stability and octahedra

Given a point of marginal stability in the moduli space where we have a good picture in terms the topological D-branes, we may ask what happens if we change $B + iJ$ slightly to move away from the point.

Suppose we consider the system of E, F and G introduced above, with $G = \mathrm{Cone}(f : E \to F)$, and move away from a point on the ms-wall. Clearly one of three things can happen:

1. $\varphi(F) - \varphi(E) - 1 = 0$ implying we remain on the ms-wall.
2. $\varphi(F) - \varphi(E) - 1 < 0$ implying that the open string becomes tachyonic. This tachyon acts like the ones studied by Sen [24] and binds E and F together to form G.
3. $\varphi(F) - \varphi(E) - 1 > 0$. This forces the grade of the other two maps in the distinguished triangle to go negative, a potential contradiction to the axioms of unitary CFT which can only be resolved if G is in fact unstable against decay to E and F [5, 10].

Thus the grades of the D-branes determine the stability constraints. This is the "Π-stability" of [9] and is the central issue we study in this paper.

The simplest proposal one could make is that a physical D-brane is an object in $\mathbf{D}(X)$ which is Π-stable against all possible decays. Note that $\mathbf{D}(X)$ is a concept from algebraic geometry and, as such, does not depend on $B + iJ$. The central charges do depend on $B+iJ$ however and so therefore do the stability conditions. One might think of the derived category data as solving some "F-term" equation of motion and the further Π-stability condition as arising from some "D-term" [5, 10].

However, this proposal suffers from a problem of self-reference which looks very problematic. In order for C to decay into A and B, both A and B must actually exist! A given triangle therefore has little information content:

- If $\varphi(B) - \varphi(A) < 1$ we cannot assert that C is stable as it may be the vertex of another triangle which allows a decay into other products.
- If $\varphi(B) - \varphi(A) > 1$ we cannot assert that C is unstable as we have not established the stability of A or B.

Conceptually, the simplest way to deal with this problem is to assume that there is some other way of determining the set of stable objects at a given "basepoint" $P_0 \in \mathscr{M}$ and then try to follow the set of stable objects as we move away from this basepoint. We define Π-stability as a criterion on the entire set of stable objects [5]. This set or list, call it $S(P)$, should be considered as a function on path space. We furthermore assign a value of φ (fixing the mod 2 ambiguity) to each object in $S(P)$.

The rules above can now be used to determine the entire list $S(P')$ at all nearby points P' given $S(P)$. To be more precise, we enumerate the following constraints, to be applied at every point P:

1. For every distinguished triangle of the form

$$\begin{tikzcd} & C \arrow[dl, "{[1]}"'] & \\ A \arrow[rr] & & B \arrow[ul] \end{tikzcd} \tag{2.16}$$

   with $A, B \in S(P)$, $\varphi(B) - \varphi(A) = 1$, we have $C \in S(P)$ with $\varphi(C) = \varphi(B)$.

2. On the side $\varphi(B) - \varphi(A) < 1$ of the ms-wall defined in 1, we have $C \in S(P)$, with all grades $\varphi(A)$, $\varphi(B)$ and $\varphi(C)$ determined by continuity.

3. Conversely, for every triangle of the form (2.16) with $A, B \in S(P)$ and $\varphi(B) - \varphi(A) > 1$, we have $C \notin S(P)$.

To get the definition started, one starts at a base point P_0 in moduli space where $S(P)$ is known *a priori*. For example, in the large volume limit, one might take $S(P)$ to be the

μ-stable coherent sheaves.[5] One might start from other points as well; an example we will discuss in more detail below is the orbifold point.

It is a very interesting question as to whether Π-stability can intrinsically determine the set of stable objects without need for other criteria. We will not attempt to address this problem in this paper.

In the examples, $S(P)$ is an infinite set, so one needs further information to use this definition in practice. For now, let us start by formulating the criterion in a way which is clearly practical, i.e. could be implemented on a computer.

Thus, let Q be a subcategory of $\mathbf{D}(X)$ consisting of a finite number of objects together with a finite number of morphisms. There are therefore a finite number of distinguished triangles composed purely of objects and morphisms in Q. This allows explicitly checking the rules above, and objects in $S(P)$ constructed in this way will be referred to as "Q-stable". Hopefully as Q is taken to be a larger and larger subset of $\mathbf{D}(X)$ we get a closer approximation to the truth of whether the corresponding D-brane really exists as a physical object. There are probably many mathematical subtleties which need to be overcome to make this hope more manifest.

The first criticism one can make of this proposal is that stability, which should be a physical notion, depends on position in the path space $\mathscr{P}$ rather than the moduli space itself. As we explore in section 4, the physical identity of an object in $\mathbf{D}(X)$ undergoes monodromy on going around singularities in $\mathscr{M}$ and so one cannot expect the notion of stability of an object in $\mathbf{D}(X)$ to depend only on the position in $\mathscr{M}$. However, one should expect stability to be a function of position in the Teichmüller space $\mathscr{T}$. That is, a loop in $\mathscr{M}$ not enclosing anything like a conifold point, Gepner point, etc., should be trivial and so the notion of stability should not change upon monodromy around this loop.

As we commented earlier, the value of φ for objects with general charge will be ambiguous on $\mathscr{T}$. However, our rules for stability only use the φ's of stable objects; indeed there is no clear physical definition of φ for an unstable object.

Grades of stable objects will be unambiguous on $\mathscr{T}$ if no path in $\mathscr{R}(A) \subset \mathscr{T}$, the region in which A is stable, encloses a zero $Z(A;P) = 0$. This will be true if we grant two conditions. First, stable objects cannot become massless at non-singular points in moduli space,

$$Z(A;P) \neq 0 \qquad \forall A \in S(P)\,. \tag{2.17}$$

Of course this is also a physical consistency condition — such a massless state would have shown up as a singularity of the conformal field theory and in the conjugate period. Second, we will assume that $\mathscr{R}(A)$ is simply connected.[6]

In the finite approximation of Q-stability, a massless object in Q which is Q-stable at a smooth point in $\mathscr{M}$ should become Q-unstable if we increase the size of Q to more

[5]Strictly speaking, this will be true only in the limit in which the volume (in string units) is much larger than any other quantity in the problem at hand (in particular, the charges Q of the objects under discussion). It is also possible that one must make some restriction on the types of singularities allowed in these sheaves.

[6]This is true in all known examples, but we should say that it is not completely obvious and it would be quite interesting if a counterexample were found.

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closely approximate real physics. We therefore define a collection Q to be "s-complete" if no Q-stable objects become massless at a (smooth) point in $\mathscr{M}$.

We now argue that s-completeness is sufficient to guarantee that the notion of Q-stability is well-defined on $\mathscr{T}$. Let us consider an object C which we declare to be stable at a given point in the path space $\mathscr{P}$. One possible decay path of C is $C \to A + B$. In figure 1 we denote by S_C a codimension one wall in $\mathscr{T}$ with $\varphi(B) - \varphi(A) = 1$. This is an ms-wall if A and B are stable.

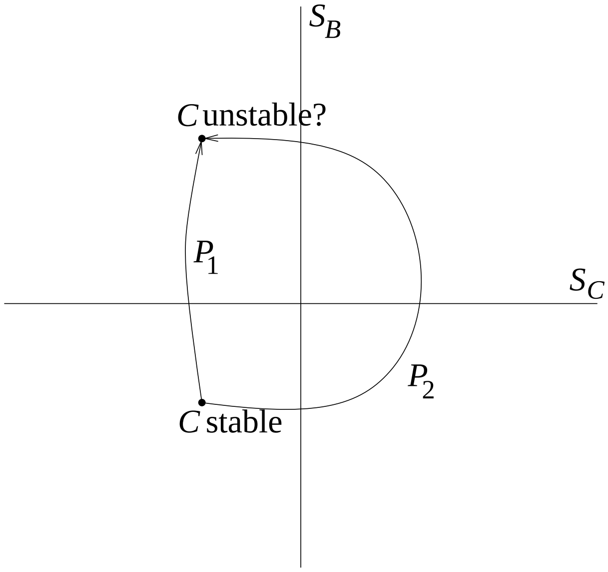

Figure 1: Competing lines of stability

Suppose there is also the possibility of B decaying into $E + F$. We denote the corresponding ms-wall by S_B. Assume B is stable to the left of this line. Let us ignore all other possible triangles (i.e., decays). Following path P_1 in the figure we see that C decays as we cross S_C. Thus the object C is unstable at the end of the path.

Consider now the path P_2 in figure 1. At the time it crosses S_C, the object B is no longer stable and so we cannot claim C necessarily decays. This could lead to the unpleasant conclusion that C is still stable if we follow the path P_2. This is exactly the situation we need to avoid in order for stability to depend only on the position in $\mathscr{T}$, since $\mathscr{T}$ essentially represents *homotopy classes* of paths.

Fortunately the "octahedral axiom" rescues us. For details of the octahedral axiom we refer to [21]. This axiom, which is obeyed by $\mathbf{D}(X)$, states a general structure associated to two distinguished triangles with a common vertex. To resolve the problem, it must be that in this situation, there always exists a third triangle predicting that if B decays, C can still decay into branes including the decay products of B. This is just what we will find.

Suppose A and B marginally bind to form C and suppose B itself is a marginally bound state of E and F. The two distinguished triangles are shown by solid arrows in figure 2. In order to get the "[1]"s at a convenient edge E and F are replaced by $E[1]$ and $F[1]$. The octahedral axiom states that given such a pair of triangles forming the "upper half" of an octahedron, there exists an object Y, unique up to isomorphism, such that two more distinguished triangles exist as shown by dotted arrows in figure 2. The octahedron thus consists of four distinguished triangles labeled at their centre by T1,...,T4 in figure 2. The four remaining faces of the octahedron commute. That is, two dotted arrows compose to give an undotted arrow etc.

Let φ denote $\varphi(A)$. One may then use (2.13) to compute the grades of the remaining 5 objects in the octahedron. These grades are shown (with $\alpha = \beta = \gamma = 0$) in the boxes in figure 2.

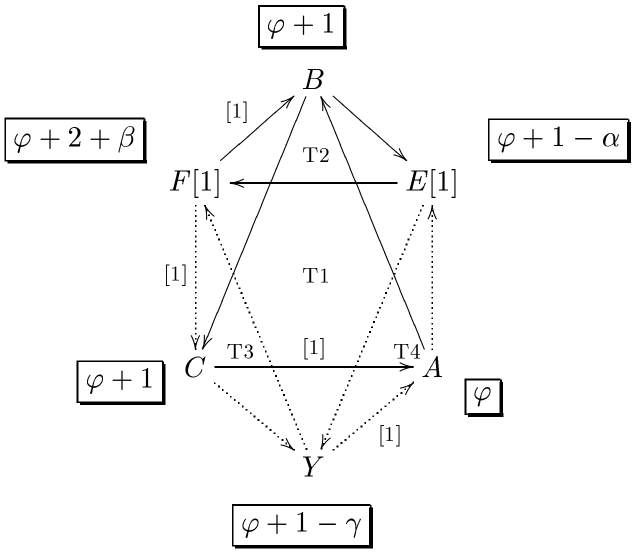

Figure 2: The octahedron for $C = A + B$ and $B = E + F$.

The triangle T3 in figure 2 thus shows that C is marginally stable with respect to a decay into F and Y (in addition to T1 giving a decay into A and B). Triangle T4 shows that Y itself is marginally stable with respect to a decay into A and E.

Now let us change $B + iJ$ a little to change some stability constraints. Suppose we preserve the marginal stability of T1 giving $C = A + B$ but we destabilize B so that it decays into $E + F$. This implies that we have arranged $\varphi(F) - \varphi(E) > 1$. Because $Z(B) = Z(E[1]) - Z(F[1])$, $\arg Z(B)$ must lie between $\arg(E[1])$ and $\arg(-F[1])$. In other words if we do not change $\varphi(A)$, and thus $\varphi(B)$, the grade of F must increase a little and the grade of E must decrease a little to make $\varphi(F) - \varphi(E) > 1$. Similarly the grade of Y will decrease a little.

We show the resulting new grades in figure 2 where α, β and γ are all positive numbers. This has an interesting effect on the triangles T3 and T4. First note that T4 gives $\varphi(E[1]) - \varphi(A) < 1$ which implies that Y is now *stable*. Second T3 gives $\varphi(F[1]) - \varphi(Y) > 1$ which implies that C is *unstable*.

We have therefore shown that if C is marginally stable against decay into A and B, but that B itself is unstable with respect to a decay into E and F, then C will always be unstable with respect to decay into F and some bound state of A and E.

Referring back to figure 1 we have shown that at the time the path P_2 crosses the line S_C, the object C must have already decayed into something else. In other words there must be another wall of marginal stability encountered before S_C. We therefore have the consistent result that C is unstable at the end of the path in figure 1 independent of the homotopy class of the path.

We give an example of this effect in section 3.5 which we hope will clarify the argument above.

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2.5 Stable sheaves and vector bundles

It is well worth comparing the notion of stability for D-branes with similar notions for vector bundles and sheaves. The familiar notion of a D-brane, before venturing into the complexities of the derived category, is that of a vector bundle E supported on X on over some subspace of X. The connection on this bundle must satisfy the hermitean-Yang-Mills equation.

A bundle E has a "slope" μ given by

$$\mu(E) = \frac{1}{\mathrm{rank}(E)} \int c_1(E) \wedge (*J) \tag{2.18}$$

where J is the Kähler form. A bundle E is said to be *stable* if it is associated to a sheaf $\mathscr{E}$ such that every coherent subsheaf $\mathscr{F}$ of lower rank satisfies $\mu(\mathscr{F}) < \mu(\mathscr{E})$. It is *semistable* if the weaker condition $\mu(\mathscr{F}) \leq \mu(\mathscr{E})$ holds.

We then have the following theorem due to Donaldson, Uhlenbeck and Yau [25, 26]

Theorem. *An bundle is stable if and only if it admits an irreducible hermitean-Yang-Mills connection. This connection is unique.*

As shown in [9] this stability condition is implied by Π-stability at large radius limit as we now review. Suppose $\mathscr{F}$ is a subsheaf of $\mathscr{E}$. We then have a short exact sequence

$$0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{G} \to 0\,, \tag{2.19}$$

for some $\mathscr{G}$ which yields a distinguished triangle in the derived category in the usual way. Near the large radius limit we may use (2.9) to show

$$\varphi(\mathscr{E}) \approx -\frac{1}{\pi} \tan^{-1} \frac{2V}{\mu(\mathscr{E})}\,, \tag{2.20}$$

where V is the volume of X. The stability condition $\mu(\mathscr{F}) < \mu(\mathscr{E})$ is then equivalent to $\varphi(\mathscr{F}) < \varphi(\mathscr{E})$. Since $ch(\mathscr{E}) = ch(\mathscr{F}) + ch(\mathscr{G})$, we have $Z(\mathscr{E}) = Z(\mathscr{F}) + Z(\mathscr{G})$ and so $\varphi(\mathscr{E})$ lies between $\varphi(\mathscr{F})$ and $\varphi(\mathscr{G})$. It follows that $\mu(\mathscr{F}) < \mu(\mathscr{E})$ implies that $\varphi(\mathscr{F}) - \varphi(\mathscr{G}[-1]) < 1$ and so $\mathscr{E} = \mathrm{Cone}(\mathscr{G}[-1] \to \mathscr{F})$ is Π-stable with respect to the triangle associated to (2.19).

Comparing this with the earlier discussion of Π-stability, the main point of difference is that the coherent sheaves form an *abelian* category: one has a well-defined notion of kernels and cokernels. The statement that "$\mathscr{F}$ is a subsheaf of $\mathscr{E}$" is another way of saying that the morphism $\mathscr{F} \to \mathscr{E}$ has no kernel and so the statement depends on the abelian structure of the category.

The derived category of coherent sheaves is not an abelian category and hence has no absolute notion of "sub-D-brane". In checking the stability of a sheaf one has to check for stability only against all subsheaves. Since a subsheaf has a lower rank, this is always a finite process. For D-branes however there is no such hierarchy of objects such as rank and therefore no analogue of a finite chain of possible decay products. Given two D-branes A and B, it is known in examples that A can decay into B for some values of the Kähler form, while B can decay into A for other values, as we will see in subsection 4.1. It is this lack of hierarchy between D-branes and their decay products which forces one to use a category which is not abelian, rather than simpler notions such as sheaves.

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2.6 Abelian subcategories and the orbifold point

The arguments we just made do not yet rule out the possibility that in *specific regions* of Teichmüller space, one can reduce the general discussion of Π-stability to stability in an abelian category, as proposed in [9]. Thus, let us explore the claim that in some region $\mathscr{R}$, the Π-stable objects are precisely those objects E which satisfy $\varphi(E') < \varphi(E)$ for all subobjects E' of E defined with respect to an abelian category $\mathscr{A}_{\mathscr{R}}$.

It is convenient to impose the further requirement that $\mathscr{A}_{\mathscr{R}}$ is a graded abelian category with integer gradings (as usual).

At present, the only known case of a point at which all relative gradings are integer, $\varphi(E) - \varphi(F) \in \mathbb{Z}$, is the orbifold point (for a $\mathbb{C}^3/\Gamma$ orbifold singularity). Near such an orbifold point, we can use conventional $\mathscr{N} = 1$ supersymmetric gauge theory to describe all bound states of fractional branes, as has been observed in many works. We now give physical arguments why this is so, to better understand in what situations this simplification might hold.

Consider a combination of fractional branes; it will fall into one of three classes. If the central charges of all the branes roughly align, then all tachyonic modes have small masses compared to the string scale, and the conditions for applicability of low energy world-volume field theory are clearly met. On the other hand, if some branes have roughly anti-aligned central charge, we can further distinguish the cases in which a fractional brane and its own antibrane are both present, and the case in which this is not so. In the first case, since the mass of the brane-antibrane tachyon is string scale and much larger than any other scale, the lowest energy configuration clearly will condense this tachyon, leading to brane-antibrane annihilation and a configuration with a simpler description (in terms of fewer fractional branes). In the second case, examination of the orbifold spectrum shows that there are no brane-antibrane tachyons present, so no possibility for a bound state involving all of the objects. Again, the simple bound states can be described using only branes or only antibranes.

This last point in the argument relies on a condition on the spectrum which holds near the orbifold point but not necessarily for a general collection of branes. Thus the condition that the constituent brane central charges roughly align is necessary but not sufficient for the applicability of supersymmetric gauge theory. The additional condition that the only string-scale tachyons are those between a brane and its own antibrane is sufficient; we have not established that it is necessary.

Having justified the use of supersymmetric gauge theory in a region $\mathscr{R}_{orb}$ around the orbifold point, we note that indeed Π-stability reduces to a stability condition in an abelian category in this region; in fact we have justified the use of θ-stability near the orbifold point as in [27], using the category of quiver representations.

We now ask what is the boundary of this region $\mathscr{R}_{orb}$. The primary constraint is that morphisms $\mathrm{Hom}(E, F)$ of negative degree can only be produced by flow from morphisms of degree zero. This is clear because we require every decay predicted by Π-stability to correspond to a decay into a subobject.

A sufficient (but perhaps not necessary) condition for this is to require that, compared to the orbifold point,

$$|\Delta\varphi(E) - \Delta\varphi(F)| < 1 \tag{2.21}$$

for all differences of gradings of objects in $\mathscr{A}_{\mathscr{R}}$.

This condition determines the region $\mathscr{R}_{orb}$ around the orbifold point in which the category of quiver representations can be used to define stability. For $\mathbb{C}^3/\mathbb{Z}_3$, its boundary (within the fundamental region) is the ms-line on which the D2-brane becomes stable. It is easily seen that this stability condition breaks down on the boundary of this region.

Within $\mathscr{R}_{orb}$, the appropriate notion of stability is a generalized notion of θ-stability, which uses the same definition of subobject, but in which the stability condition $\theta \cdot n > \theta \cdot n'$ for every subobject $E' \to E$ is replaced by the condition $\varphi(E') < \varphi(E)$. The hierarchy between branes and decay products is governed by the total number of fractional branes, which is always decreasing in subobjects; this number is determined by the charge of an object.

For more general (compact) Calabi-Yau's, there probably does not exist any point in moduli space at which all brane central charges roughly align or antialign. Such points will exist for subcategories of branes carrying a subset of the possible charges, and it would probably be quite useful to analyze the spectrum within these subcategories along the lines which we just described. Let us however continue to discuss proposals which could describe all the branes.

A logical extension of the considerations in this subsection is to try to find a set of regions $\mathscr{R}_i$ whose union covers the Teichmüller space, and corresponding graded categories $\mathscr{A}_i$ each appropriate in a region $\mathscr{R}_i$, meaning that within $\mathscr{R}_i$ all decays predicted by Π-stability in fact reduce to decays into subobjects as defined within $\mathscr{A}_i$. If such a picture can be made to work, it would be reasonable to refer to the regions $\mathscr{R}_i$ as "phases," as the choice of $\mathscr{A}_i$ already tells us quite a lot of information about the BPS spectrum.

The simplest proposal for constructing such an abelian category $\mathscr{A}_i$ goes as follows. First, one needs to postulate integer gradings for all of the objects. The simplest prescription one can make is to take the gradings $\varphi(E)$ at a specific point P in moduli space and round them off to the nearest integer; call this $n(E)$. As long as we make a consistent choice of $n(E)$ for each E, any choice of rounding will work. This choice then leads to a definition of integer grading for each morphism; let $n(f)$ be the grading of the morphism f.

As in the orbifold example, we want to arrange things so that all decays predicted by Π-stability in fact proceed through subobjects as defined by exact sequences involving morphisms of grading $n = 0$. This condition will again define the region $\mathscr{R}$ in which a given choice of integer gradings can be used. A potential difficulty with this idea appears at this point, as there is no guarantee that the region $\mathscr{R}$ so defined will have finite extent.

Assuming it does, one then wants to show that this data indeed defines an abelian category; in particular that morphisms have kernels and cokernels. As mentioned in [5], the appropriate framework for this discussion is the formalism of t-structures; we hope to return to this in future work.

3. Examples on the quintic

The analysis of section 2 is clarified in this section by analyzing various examples of Π-stability in the context of the quintic hypersurface in $\mathbb{P}^4$.

3.1 Basics of the moduli space and periods

In this section we review the techniques required from Candelas et al [1]. Let X be the quintic hypersurface in $\mathbb{P}^4$. The mirror Y is then a $(\mathbb{Z}_5)^3$ orbifold of a quintic with general defining equation

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\psi x_0 x_1 x_2 x_3 x_4\,. \tag{3.1}$$

The single complex variable ψ spans the one-dimensional moduli space of complex structures of Y. This encodes the mirror of the one-dimensional moduli space of complexified Kähler forms on X. It is useful to introduce the variable $z = (5\psi)^{-5}$.

The Gepner point of the moduli space of X is mirror to $\psi = 0$. The large radius limit of X is mirror to $z = 0$. At $\psi = 1$, one has the "conifold" point where Y becomes singular (acquiring a conifold point). This is mirror to the fact that the conformal field theory on X becomes singular for the corresponding value of $B + iJ$.

Since $\dim H_3(Y) = 4$, we have 4 independent periods $\int_\Gamma \Omega$ for the holomorphic 3-form Ω. These periods satisfy the Picard-Fuchs equation (see [28] for a nice account of this). Following [1] we use the following periods given as a power series around the Gepner point:

$$\varpi_j = -\frac{1}{5}\sum_{m=1}^{\infty} \frac{\alpha^{(2+j)m}\Gamma(m/5)}{\Gamma(m)\Gamma(1-m/5)^4} z^{-m/5}\,. \tag{3.2}$$

We may use $\varpi_0, \ldots, \varpi_3$ as a basis.[7]

One may then analytically continue these periods using Barnes' integral methods to match this basis with series around the large radius limit. We use the following basis near $z = 0$:

$$\begin{aligned}
\Phi_0 &= \frac{1}{2\pi i}\int \frac{\Gamma(5s+1)\Gamma(-s)}{\Gamma(s+1)^4}\left(e^{\pi i}z\right)^s ds \\
&= \sum_{n=0}^{\infty}\frac{(5n)!}{n!^5} z^n \\
&= -\frac{1}{5}\sum_{m=1}^{\infty}\frac{\alpha^{2m}\Gamma(m/5)}{\Gamma(m)\Gamma(1-m/5)^4} z^{-m/5} \\
&= \varpi_0\,.
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
\Phi_1 &= -\frac{1}{2\pi i}\cdot\frac{1}{2\pi i}\int \frac{\Gamma(5s+1)\Gamma(-s)^2}{\Gamma(s+1)^3} z^s\, ds \\
&= \frac{1}{2\pi i}\log z + O(z) = t + O\left(e^{2\pi i t}\right) \\
&= -\frac{1}{5}(\varpi_0 - 3\varpi_1 - 2\varpi_2 - \varpi_3)\,.
\end{aligned} \tag{3.4}$$

[7] In this section and section 4, we follow the convention that the relation (2.10) between $\varphi(E)$ and $Z(E)$ has an additional minus sign, and $Z(E)$ is the complex conjugate of that defined in (2.9).

$$\begin{aligned}
\Phi_2 &= -\frac{1}{2\pi^2}\cdot\frac{1}{2\pi i}\int \frac{\Gamma(5s+1)\Gamma(-s)^3}{\Gamma(s+1)^2}\left(e^{\pi i}z\right)^s\,ds \\
&= -\frac{1}{4\pi^2}(\log z)^2+\frac{1}{2\pi i}\log z-\frac{5}{6}+O(z) \\
&= t^2+t-\frac{5}{6}+O\left(e^{2\pi i t}\right) \\
&= \frac{2}{5}(-2\varpi_0+\varpi_2+\varpi_3)\,. \qquad (3.5) \\
\Phi_3 &= \frac{1}{(2\pi i)^3}\cdot(-6)\cdot\frac{1}{2\pi i}\int \frac{\Gamma(5s+1)\Gamma(-s)^4}{\Gamma(s+1)}z^s\,ds \\
&= \frac{i}{8\pi^3}(\log z)^3+\frac{7i}{4\pi}\log z-\frac{30i\zeta(3)}{\pi^3}+O(z) \\
&= t^3-\frac{7}{2}t-\frac{30i\zeta(3)}{\pi^3}+O\left(e^{2\pi i t}\right) \\
&= -\frac{6}{5}(2\varpi_1+2\varpi_2+\varpi_3)\,. \qquad (3.6)
\end{aligned}$$

These analytic continuations are valid for

$$-2\pi < \arg z < 0. \qquad (3.7)$$

In each case above the integral is taken along the path corresponding to a straight line from $s=\epsilon-i\infty$ to $s=\epsilon+i\infty$, where $0<\epsilon<1$. The variable t above corresponds to $B+iJ=te_1$ for X where e_1 generates $H^2(X,\mathbb{Z})$. The "mirror map" determines t uniquely as

$$t=\frac{\Phi_1}{\Phi_0}\,. \qquad (3.8)$$

We refer to [29], for example, for an account of the history and status of the mirror map.

The above results show that the true stringy complexified Kähler moduli space $\mathscr{M}$ of X is the $\mathbb{P}^1$ with affine coordinate z; the periods are singular or have branch points at the three points mentioned above. (The Gepner point is only an orbifold singularity, and physics is nonsingular there.)

As discussed in section 2.4 the stability questions are best thought of in terms of the Teichmüller space rather than the moduli space. In terms of the "presentation of data" we wish to give in this example, we basically have three choices of how to parametrize the Kähler moduli space:

1. We can use z (or ψ) as above as a parameter on the moduli space. Since z is not a parameter on the Teichmüller space, the periods and questions of stability will not be a single valued function of z.

2. We can use the theory of automorphic functions to construct the Teichmüller space as the upper half-plane as was done in section 2.2 of [1]. This has the advantage of presenting the periods as single valued functions. The disadvantage of this approach is that this Teichmüller space appears to be remarkably unphysical. For example, the real line of the upper half-plane will contain a dense set of copies of both the

conifold point and the large radius limit point. The former are a finite distance from a generic point in the moduli space while the latter are an infinite distance! This is tied in with problems of notions of T-duality for such string compactifications [30].

3. We can use $t = B + iJ$ as the parameter. This is the most physical parameter for the moduli space. The disadvantage of this approach is that fundamental regions of the moduli space do not "tile" the t-plane leading to multivaluedness as in the z-plane. This problem should be clear from the figures in section 3.

The choice we make in this paper is to go with the third option and use t to plot lines of marginal stability. This is viable because the lines of marginal stability we consider usually remain in the "Calabi-Yau" phase, i.e., remain on the large radius side of the conifold point. In this phase, t is certainly the most natural parameter. Some of the ms-lines analyzed later in this section venture outside this phase but they do not do unpleasant things such as crossing themselves when plotted in the t-plane. If such pathologies occur, as they surely do in more complicated examples, it might be easier to visualize the ms-lines in the less physical choice of using the upper half-plane as the Teichmüller space.

3.2 The exact computation of φ

For a Calabi-Yau threefold, an A-type D-brane corresponds to a 3-cycle. The central charge, Z, of this D-brane is then given, up to some normalization, exactly by the period associated to this cycle. The normalization will not concern us since we are only interested in differences in gradings and hence ratios of periods. We may use this fact, together with the knowledge of section 3.1, to determine the exact central charges of B-type branes on the quintic threefold.

Given a particular element of $\mathbf{D}(X)$ we may compute the Chern character using

$$ch(\mathscr{E}^\bullet) = \sum_n (-1)^n \, ch(\mathscr{E}^n) \,. \tag{3.9}$$

Equation (2.9) then determines the large radius behaviour of $Z(\mathscr{E}^\bullet)$ as a series in t. This is then enough to fix the exact linear combination of ϖ_j's from section 3.1 which represents the period on Y and hence the exact form of $Z(\mathscr{E}^\bullet)$.

In order to perform this analysis we need to know precisely what constitutes the "quantum corrections" of (2.9). These corrections come from two sources:

1. Nonperturbative corrections: These are of the form $a_n \exp(2\pi i n t)$ for $n > 0$.

2. Perturbative corrections: These are of the form $b_m t^m$. Since these corrections begin at "four loops" [31], the highest power of t appearing will be three less than the leading order ("one loop") term.

Note therefore that the only effect of the perturbative corrections is to contribute a constant term in (3.6). We claim that this constant term acts to cancel the $30i\zeta(3)/\pi^3$ term in (3.6). One way to justify this is to do a direct computation of the 4-loop term which we have not done.

Instead let us use the fact the the singularity in the conformal field theory associate to X at the "conifold point" is associated to a D-brane becoming massless. The monodromy of the periods around the conifold point given by table 3.1 of [1] immediately implies that the period associated to this D-brane must be $-\varpi_0 + \varpi_1$. The analysis of the previous section shows that at large radius this period behaves as

$$\frac{5}{6}t^3 + \frac{25}{12}t - \frac{25i\zeta(3)}{\pi^3}\,. \tag{3.10}$$

Using (2.9) and $td(T_X) = 1 + \frac{5}{6}e_1^2$ for the quintic we have

$$ch(\mathscr{E}^\bullet) = 1 + \left(c - \frac{25i\zeta(3)}{\pi^3}\right) e_1^3\,, \tag{3.11}$$

where c represents the 4-loop correction. Since c is imaginary (see [1] for example) the reality of ch confirms our assertion.

Furthermore we see immediately that the D-brane that vanishes at the conifold point satisfies $ch(\mathscr{E}^\bullet) = 1$. It is therefore natural to associate it with the structure sheaf $\mathscr{O}$ as asserted in [32].

Now that we have fixed this 4-loop term we have an exact method for associating a linear combination of ϖ_j's (over the rational numbers) with the central charge of any element of $\mathbf{D}(X)$. We therefore have an exact method of computing the grade φ of any B-type D-brane on the quintic. This method should generalize to any example where the Picard-Fuchs system is understood in the language of generalized hypergeometric functions.

In the examples below we computed φ numerically by integrating the Picard–Fuchs equation with a Runge–Kutta algorithm. The initial conditions were specified by using the power series expansions of ϖ_j (3.2).

3.3 N D4-branes

As a first example consider the following map

$$\mathscr{O}(-N) \xrightarrow{f(x_i)} \mathscr{O}\,, \tag{3.12}$$

where $f(x_i)$ is a polynomial of degree N in the homogeneous coordinates of $\mathbb{P}^4$. By $\mathscr{O}(-N)$ we mean the restriction of the line bundle on $\mathbb{P}^4$ with $c_1 = -N$ to the quintic X. If $N = 1$ then the short exact sequence

$$0 \longrightarrow \mathscr{O}(-1) \xrightarrow{f(x_i)} \mathscr{O} \longrightarrow \mathscr{O}_D \longrightarrow 0\,, \tag{3.13}$$

implies that the cone of the map (3.12) is simply $\mathscr{O}_D$. Here D is the divisor 4-cycle in X given by the zeroes of $f(x_i)$ and $\mathscr{O}_D$ is its structure sheaf extended by zero over X. That is $\mathscr{O}_D$ is the sheaf representing the 4-brane wrapped on D.

Similarly for general N we have a collection of N 4-branes on X. If $f(x_i)$ takes on particular forms we may make these 4-branes coalesce or not as we wish.

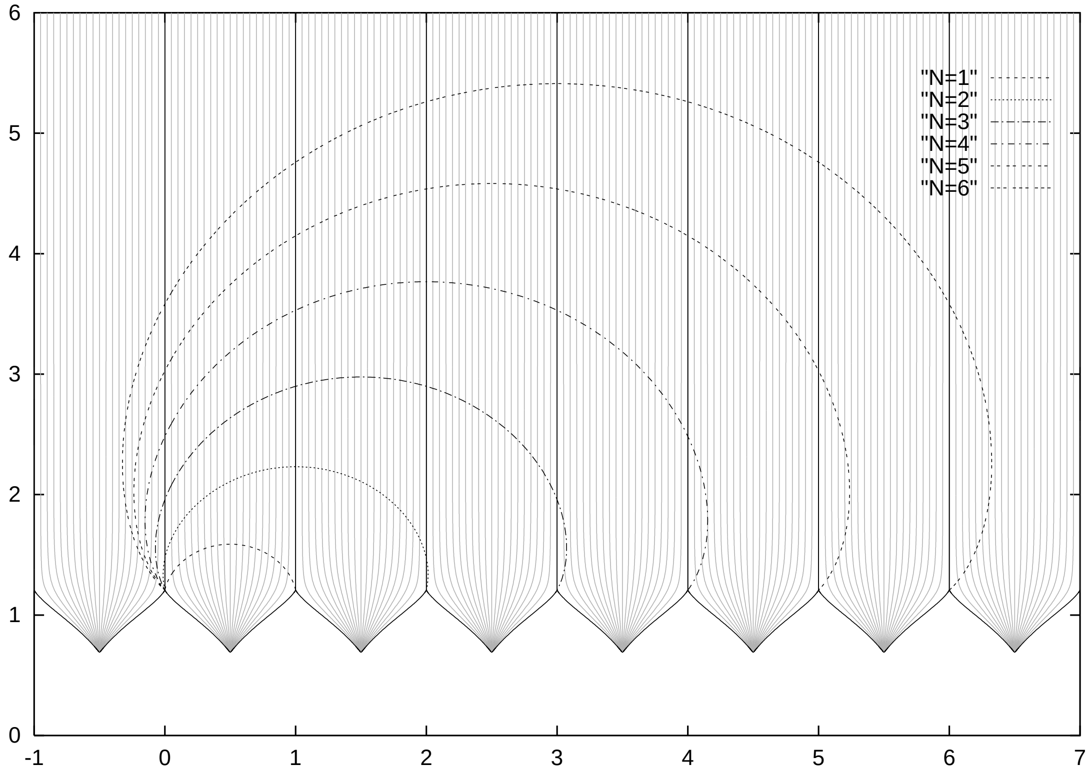

Figure 3: Stability of N D4-branes in the t-plane.

The methods of section 3.2 can be used to determine

$$Z(\mathscr{O}(-N)) = -\frac{1}{6}\left(5N^3 + 3N^2 + 16N + 6\right)\varpi_0 + \frac{1}{2}\left(3N^2 + 3N + 2\right)\varpi_1 + N^2\varpi_2 + \\ + \frac{1}{2}N(N-1)\varpi_3\,. \tag{3.14}$$

A collection of N 4-branes is stable against decay to the 6-brane $\mathscr{O}$ and the anti-6-brane with 4 brane charge $-N$ if

$$\varphi(\mathscr{O}) - \varphi(\mathscr{O}(-N)) < 1\,. \tag{3.15}$$

The resulting lines of marginal stability for various values of N are shown in figure 3. The D4 branes are stable *above* the line (i.e., large radius) as one would expect. Note that this picture is of a series of fundamental regions in the t-plane each spanning a shift in the B-field (i.e., real part of t) by one. The starting region, as required by (3.7), is the one where $-1 < B < 0$. The regions to the right are obtained by analytic continuation.

Note that the lines of marginal stability have one end at the conifold point where $Z(\mathscr{O})$ vanishes and the other end is at the copy of the conifold point where $Z(\mathscr{O}(-N))$ vanishes. This is a general feature of the lines of marginal stability. They can only end at a point where $\arg(Z)$ becomes ill-defined, i.e., where one of the periods vanishes.

Note that for large N one may use (2.9) directly to show that asymptotically the line of stability becomes an arc of a circle of radius $N/\sqrt{3}$ with centre $t = N/2 + iN/2\sqrt{3}$.

3.4 Some exotic D-branes

As discussed in [10] one may use the D4-brane construction in section 3.3 as a basis of a construction of more exotic objects. The map $f(x_i)$ in (3.12) lives in the space $\mathrm{Hom}(\mathscr{O}(-N),\mathscr{O}) = \mathrm{Hom}(\mathscr{O},\mathscr{O}(N)) = \mathrm{Ext}^0(\mathscr{O},\mathscr{O}(N)) = H^0(\mathscr{O}(N))$ (see, for example, [33]). By Serre duality, $H^0(\mathscr{O}(N))$ is dual to $H^3(\mathscr{O}(-N)) = \mathrm{Ext}^3(\mathscr{O},\mathscr{O}(-N))$. Thus every example of a map f has some "dual" map $f^\vee \in \mathrm{Ext}^3(\mathscr{O},\mathscr{O}(-N))$.

In terms of the derived category, an element of $\mathrm{Ext}^3(\mathscr{O},\mathscr{O}(-N))$ is a morphism

$$\underline{\mathscr{O}} \xrightarrow{f^\vee} \underline{\mathscr{O}(-N)}[3]\,, \tag{3.16}$$

where, recalling the notation of section 2.1, $\underline{\mathscr{O}(-N)}[3]$ refers to a complex whose only nontrivial entry is $\mathscr{O}(-N)$ at position -3. The cone of this map $\mathscr{X}_N = \mathrm{Cone}(f^\vee : \underline{\mathscr{O}} \to \underline{\mathscr{O}(-N)}[3])$ is an object in $\mathbf{D}(X)$ we wish to study in this section. Note that we may deform $\mathscr{X}_N$ into a whole family of D-branes by varying $f^\vee$ — just as 4-branes may be deformed by varying f.

The rules of section 2 require that $\varphi(\mathscr{E}[n]) = \varphi(\mathscr{E}) + n$. It follows that the condition for stability of $\mathscr{X}_N$ is that

$$3 + \varphi(O(-N)) - \varphi(O) < 1\,. \tag{3.17}$$

This condition can be analyzed in the same way as the previous section and the result is shown in figure 4. This time the region of stability lies *below* the marginal line. That is, these exotic objects only exist at small radius. Clearly something odd has happened for $N = 2$ which is the subject of the next section.

As before, for large N one may use (2.9) directly to show that asymptotically the line of stability becomes an arc of a circle of radius $N/\sqrt{3}$ whose centre this time is located at $t = N/2 - iN/2\sqrt{3}$.

It is perhaps worth emphasizing that our discussion of the exotic objects $\mathscr{X}_N$ requires φ not be to confined to an interval of width 2. The stability of the $\mathscr{X}_N$'s below the line of marginal stability occurs precisely because the relative grades of the sheaves $\mathscr{O}$ and $\mathscr{O}(-N)$ exceeds 2!

3.5 The peculiar case of $\mathscr{X}_2$

The first thing to notice about the $N = 2$ line in figure 4 is that we have used an inappropriate set of fundamental regions. The line of marginal stability passes outside the fundamental region $0 < B < 1$ *below* the conifold point. It is more appropriate the use Candelas et al's "scorpion" [1] as shown in figure 5. This consists of the 5 fundamental domains around a Gepner point at $t = 1/2 + \frac{i}{2}\cot\pi/5$.

The marginal line for $N = 2$ is shown as "T1" in figure 5 for reasons to become clear shortly. The next striking feature of this line is that it appears to end to the right at a point of no obvious significance. This point is where the period of $\mathscr{O}(-2)$ vanishes but it does not correspond to a conifold point or any other point where the conformal field theory may diverge. If a physical D-brane becomes massless, the associated conformal field theory must diverge. It must therefore be that the D-brane associated to $\mathscr{O}(-2)$ (and hence $\underline{\mathscr{O}(-2)}[3]$) becomes unstable as we move rightwards along the line.

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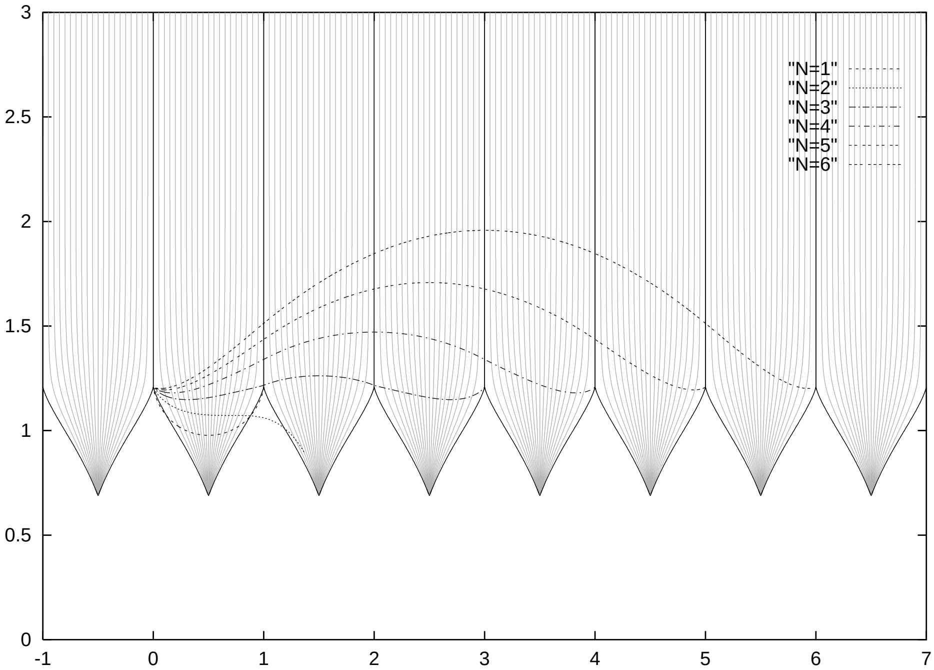

Figure 4: Stability of the exotic objects $\mathscr{X}_N$ in the t-plane.

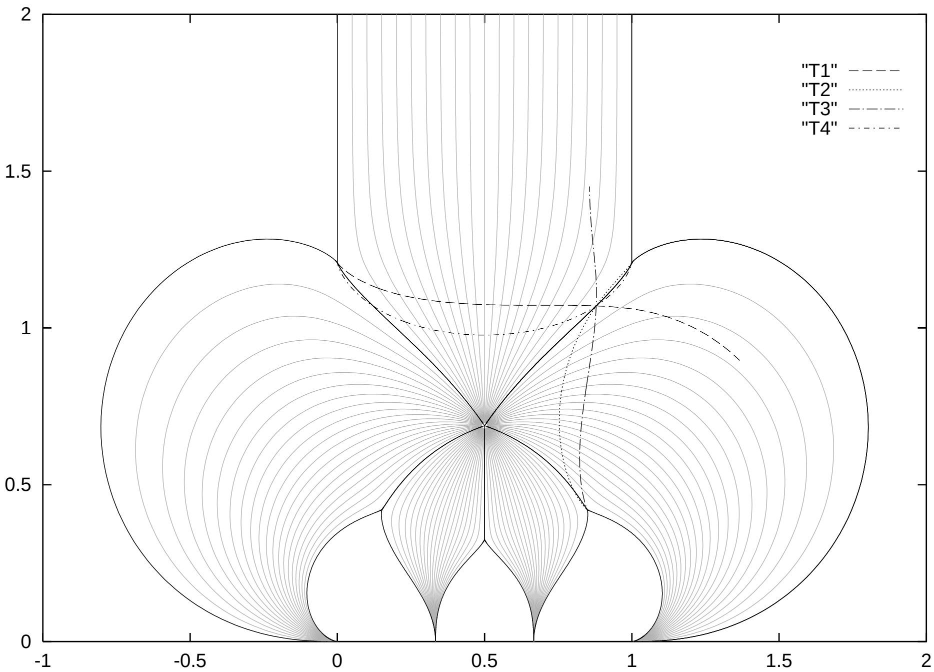

Figure 5: Lines of stability for the case of $\mathscr{X}_2$.

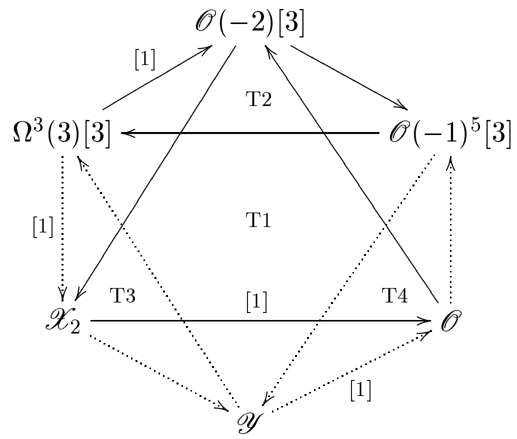

Figure 6: The octahedron for $\mathscr{X}_2$.

We are now in the situation discussed in section 2.4. In analyzing the stability of $\mathscr{X}_2$ we need to concern ourselves with the stability of one of its decay products $\underline{\mathscr{O}(-2)}[3]$. One possible decay mode of $\underline{\mathscr{O}(-2)}[3]$ which fits in nicely with this picture is as follows.

On $\mathbb{P}^4$ the following sequence is exact for $0 \leq n \leq 5$:

$$0 \to \Omega^n(n) \to \mathscr{O}^{\oplus\binom{5}{n}} \to \mathscr{O}(1)^{\oplus\binom{5}{n-1}} \to \ldots \to \mathscr{O}(n) \to 0\,, \tag{3.18}$$

where Ω^n is the nth exterior power of the cotangent sheaf. Putting $n = 3$ and $n = 5$ into this formula and comparing results we obtain the short exact sequence

$$0 \to \mathscr{O}(-2) \to \mathscr{O}(-1)^{\oplus 5} \to \Omega^3(3) \to 0\,. \tag{3.19}$$

As these sheaves are locally free, this sequence remains exact when we restrict to the quintic hypersurface X. We implicitly perform this restriction from now on.[8] This short exact sequence gives a distinguished triangle with $\underline{\mathscr{O}(-2)}[3]$ at one vertex as desired. There is good reason to believe that both $\mathscr{O}(-1)$ and $\Omega^3(3)$ are stable in the region we are considering. This is because they are the so-called "fractional branes" of [34]. When these branes become massless they are responsible for the singularities of the conformal field theories associated to the corresponding conifold points in figure 5.

The relevant octahedron is shown in figure 6 (where we omit various underlines for clarity). The triangle "T1" is the original triangle defining $\mathscr{X}_2$, and "T2" comes from (3.19). "T4" now defines the D-brane $\mathscr{Y}$ and determines its stability. Finally "T3" allows for a decay of $\mathscr{X}_2$ into $\mathscr{Y}$ and $\underline{\Omega^3(3)}[3]$. The corresponding four lines of marginal stability are shown in figure 5. Note that the analysis of section 2.4 shows in general that the point of intersection of T1 and T2 will also lie on T3 and T4.

Now we have a consistent picture for the stability of $\mathscr{X}_2$. It is stable (at least with respect to the triangles we have considered) *below* the T1 line and to the *left* of the T3 line. The rightmost point of the T1 line, and similarly the top of the T3 line, have no physical significance as $\mathscr{X}_2$ decays before they can be reached.

[8]Note in particular that Ω refers to the cotangent sheaf of $\mathbb{P}^4$ restricted to X — not the cotangent sheaf of X!

4. Monodromy

4.1 A point on the quintic

We have seen above that the properties of a D-brane are not purely a function of the moduli space but rather the universal cover of the moduli space, or the Teichmüller space. The case of $\mathscr{O}(-2)$ clearly illustrates this. In section 3.3 we saw how $\mathscr{O}(-2)$ becomes massless at the conifold point where $B = 2$. In section 3.5 we saw it instead becomes massless at a point in the interior of the moduli space.

Of course, two fundamental regions of the moduli space must represent identical physics. The set of D-branes and their physical properties is determined completely by a point in the moduli space. It can only be a given D-brane's interpretation as an element of $\mathbf{D}(X)$ that depends on how we got there, or equivalently, by a point in the Teichmüller space.

This leads immediately to the conclusion that monodromies in the moduli space induce *autoequivalences* of $\mathbf{D}(X)$. Such an autoequivalence is always induced by a Fourier–Mukai transform [35]. Fixing notation, given projections

$$\begin{tikzcd} & X \times X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & X \end{tikzcd} \tag{4.1}$$

an object $\Psi \in \mathbf{D}(X \times X)$ induces a transform

$$T_\Psi(A) = \mathbf{R}p_{2*}\left(\Psi \overset{\mathbf{L}}{\otimes} p_1^* A\right), \tag{4.2}$$

for any object A of $\mathbf{D}(X)$.

Based on the action on the Chern characters, Kontsevich [36] conjectured that for the monodromy around the *conifold* point in the case of the quintic, Ψ is given by

$$\ldots \to 0 \to \mathscr{O}_{X\times X} \to \underline{\mathscr{O}_\Delta} \to 0 \to \ldots \tag{4.3}$$

where $\Delta \subset X \times X$ is the diagonal embedding of X, and the map in (4.3) is the obvious restriction map. Seidel and Thomas [37] analyzed many aspects of this transform in the context of "homological mirror symmetry". Indeed they outlined an interesting framework in this context in which one can motivate Kontsevich's conjecture. (Note that we will not directly appeal to this conjectured strong form of mirror symmetry in this paper.)

Note that (4.2) can be written in a simpler form involving only $\mathbf{D}(X)$:

$$T_\Psi(A) = \operatorname{Cone}(e : \operatorname{Hom}(\underline{\mathscr{O}}, A) \otimes \underline{\mathscr{O}} \to A), \tag{4.4}$$

where e is the evaluation map. We refer to [37] for the precise definitions of the manipulations in (4.4).

We are now in a position to justify Kontsevich's proposal at the level of the derived category, rather than just the cohomology class of Chern characters. We begin with the 0-brane $A = \underline{\mathscr{O}_p}$, i.e., the skyscraper sheaf at a point $p \in X$ at zeroth position in the complex.

Let $\mathscr{I}_p$ denote the ideal sheaf of functions vanishing at p. The short exact sequence

$$0 \to \mathscr{I}_p \to \mathscr{O} \to \mathscr{O}_p \to 0\,, \tag{4.5}$$

induces the distinguished triangle

$$\begin{tikzcd} & \underline{\mathscr{I}_p}[1] \arrow[dl, "{[1]}"', "w"] & \\ \underline{\mathscr{O}} \arrow[rr, "u"] & & \underline{\mathscr{O}_p}. \arrow[ul, "v"'] \end{tikzcd} \tag{4.6}$$

The grades of these objects can be calculated as before given the Chern characters

$$\begin{aligned} ch(\mathscr{O}) &= 1 \\ ch(\mathscr{O}_p) &= \frac{1}{5}e_1^3 \\ ch(\mathscr{I}_p) &= 1 - \frac{1}{5}e_1^3\,. \end{aligned} \tag{4.7}$$

Begin at a point near large radius limit and assume $-1 < B < 0$, i.e., $0 < \arg\psi < 2\pi/5$. Now loop clockwise around the conifold point at $\psi = 1$ as shown in figure 7. The straight lines in this figure denote the boundaries of fundamental regions of the moduli space along $\arg\psi = 2\pi n/5$. The same path is shown in figure 8 in the t-plane.

At large radius limit, when we begin our journey, it is easy to show that $\varphi(\mathscr{O}_p) - \varphi(\mathscr{O}) = 3/2$. Therefore the string on side u of (4.6) is massive and so $\underline{\mathscr{I}_p}[1]$ is unstable. This is a quick way of showing the instability of the D0-D6 system as discussed in [38, 39].

The first interesting thing that happens as we follow the path is that the string on side u becomes tachyonic thus stabilizing $\underline{\mathscr{I}_p}[1]$. This line of marginal stability is labeled "ideal" in figures 7 and 8. The ideal sheaf is stable to the *left* of this line in figure 7.

Continuing along the path the next interesting event occurs when the string along side w of (4.6) becomes massive thus destabilizing the 0-brane $\underline{\mathscr{O}_p}$. This is exactly the instability in the 0-brane anticipated in [6]. The line of marginal stability is shown as "point" in figures 7 and 8. The 0-brane is stable to the *left* of this line.

The net result is that our set of stable D-branes has gained $\underline{\mathscr{I}_p}[1]$ and lost $\underline{\mathscr{O}_p}$. It is not hard to see that this course of events is entire due to the fact that the period associated to $\mathscr{O}$ had a simple zero at the conifold point. That is, it is intrinsically associated to monodromy around the conifold point.

It is therefore natural to assert that monodromy around the conifold point should be associated to *replacing* the 0-brane $\underline{\mathscr{O}_p}$ by the ideal sheaf $\underline{\mathscr{I}_p}[1]$. This is exactly the result predicted by (4.4) since

$$\begin{aligned} T_\Psi(\underline{\mathscr{O}_p}) &= \left(\mathscr{O} \to \underline{\mathscr{O}_p}\right) \\ &= \underline{\mathscr{I}_p}[1]\,. \end{aligned} \tag{4.8}$$

Note that this example provides the type of phenomenon promised in section 2.5. At one point in the moduli space the object $\underline{\mathscr{I}_p}[1]$ decays into $\underline{\mathscr{O}_p}$ and $\underline{\mathscr{O}}$ whereas elsewhere in the moduli space $\underline{\mathscr{O}_p}$ decays into $\underline{\mathscr{I}_p}[1]$ and $\underline{\mathscr{O}}$. This shows the "nonabelianness" of the category of D-branes.

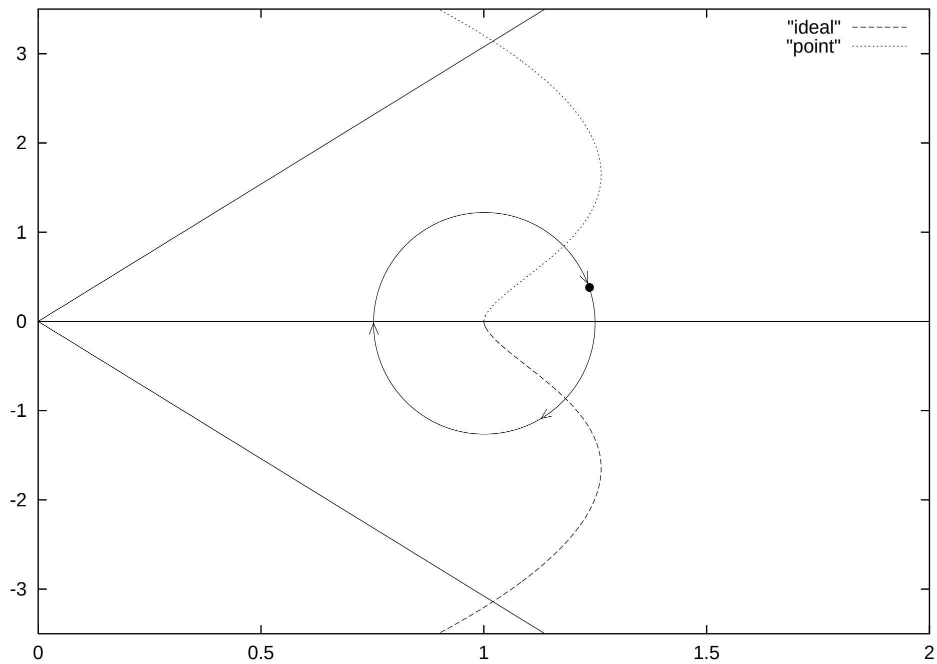

Figure 7: Lines of stability for skyscrapers and ideals in the ψ-plane.

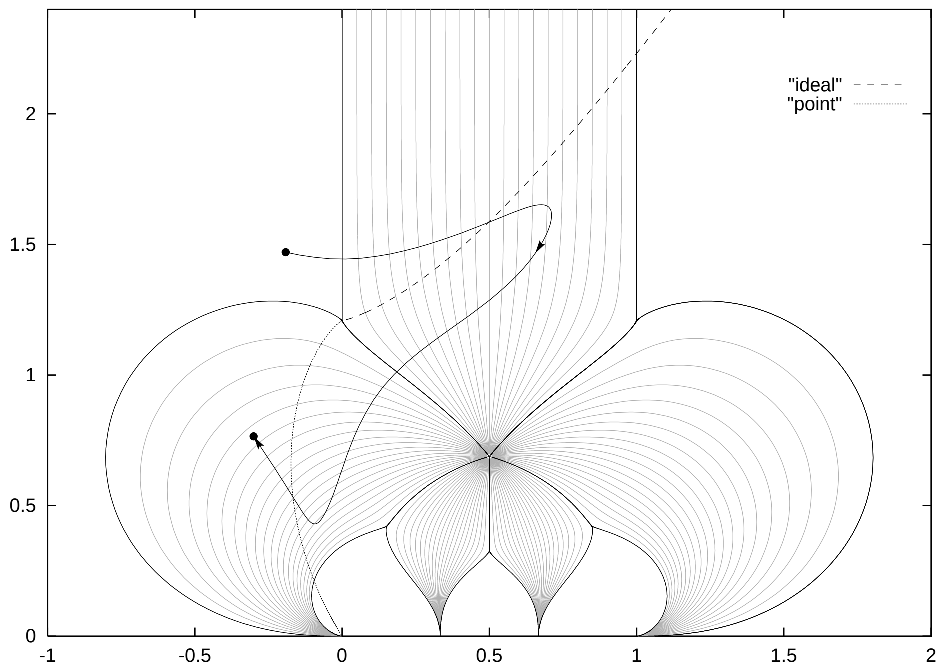

Figure 8: Lines of stability for skyscrapers and ideals in the t-plane.

This example displays an important difference from the behavior of the BPS spectrum in pure SU(2) SYM (and many other examples), as described in [40, 41, 42]. The analogy is of course that both problems exhibit ms-lines passing through a point at which a single BPS state goes massless, on which generic objects with the conjugate charge decay. The difference is that in SYM, this part of the spectrum is smaller in the strong coupling regime; the W boson and generic dyons decay in this regime, leaving a single stable massive conjugate brane (with its antibrane). In the present example, the spectrum is larger in the stringy (strongly coupled sigma model) regime. In particular, on the left of the ms-lines in figure 7, *both* the original D0 $\mathscr{O}_p$ *and* its monodromy image $\mathscr{I}_p[1]$ are stable. This is a simple consequence of the different behaviors of the central charges in the two problems, but shows that overly simple generalizations about the qualitative behavior of the spectrum can fail.

This example is also interesting from the point of view of "D-geometry" as discussed in [43]. One of the prototypical problems one can study to get a handle on stringy geometry is the moduli space metric of the D0-brane; this D0-metric is a canonically defined metric which for Calabi-Yau exhibits the well-known corrections to Ricci-flatness [31], assuming that the D0 exists in the stringy regime. What we have found is that the D0 not only (almost certainly) exists,[9] but that in the stringy regime there are two candidate D0's, which can be obtained by going around either side of the conifold point, leading to two D0-metrics. It would be interesting to know if these are necessarily the same or are related in some simple way.

4.2 The general case

Given the above example we can try to generalize to the case of any object in $\mathbf{D}(X)$ for the quintic threefold X. We know that the monodromy must induce an autoequivalence on $\mathbf{D}(X)$ and must therefore arise as a Fourier–Mukai transform associated to some $\Psi \in \mathbf{D}(X \times X)$.

Putting $A = \underline{\mathscr{O}_p}$ in (4.2) yields

$$\begin{aligned} T_\Psi(\underline{\mathscr{O}_p}) &= \mathbf{R}p_{2*}\left(\Psi|_{p\times X}\right) \\ &= \Psi|_{p\times X}\,, \end{aligned} \tag{4.9}$$

where $|_{p\times X}$ is the derived category version of the restriction map (i.e., $\overset{\mathbf{L}}{\otimes}(\underline{\mathscr{O}_p}\boxtimes\underline{\mathscr{O}})$). We therefore have that

$$\Psi|_{p\times X} = \left(\mathscr{O} \to \underline{\mathscr{O}_p}\right). \tag{4.10}$$

The most natural and obvious solution to this equation for Ψ is that

$$\Psi = \left(\mathscr{O}_{X\times X} \to \underline{\mathscr{O}_\Delta}\right) \tag{4.11}$$

as predicted by Kontsevich. We will establish this as the only possibility shortly.

[9] Although we did not consider decays into states with D2 or D4 charge, this is implausible for a variety of reasons, the simplest being that such states which are known to exist have larger mass than the D0 near the conifold point. The stability of the 0-brane has also been anaylzed in [44].

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The argument we gave in section 4.1 generalizes directly to any object E satisfying

$$1 = \sum_i \dim \mathrm{Hom}\,(\mathscr{O}, E[i])\,. \tag{4.12}$$

Given that there is a unique morphism f between $\mathscr{O}$ and E in some degree, there is a unique triangle involving these two objects and $\mathrm{Cone}(f)$ which will play the role of (4.5). Following the loop around the conifold point will again stabilize $\mathrm{Cone}(f)$ and destabilize E, again motivating the claim that (4.4) is the monodromy image of E.

Note that $\mathscr{O}$ itself is not actually invariant upon moving around the conifold point. One can show that

$$T_\Psi(\underline{\mathscr{O}}) = \underline{\mathscr{O}}[-2]\,, \tag{4.13}$$

consistent with the claim that the combination of this FMT and flow of gradings is a physical invariance.

The same argument predicts that any object E with non-zero $\mathrm{Hom}(\mathscr{O}, E[i])$, not necessarily satisfying (4.12), will decay, but now one does not get a unique candidate to replace it. Besides the evaluation map e used in (4.4), one will in general form other objects as bound states of E and $\mathscr{O}$ which are cones of other (non-canonical) maps, such as the individual elements of $\mathrm{Hom}(\mathscr{O}, E[i])$ or other direct sums of these. While adiabatically varying the moduli around the conifold point should lead to a unique decay process $E \to E' + n\mathscr{O}$, it is not completely clear to us that E' must be the monodromy image in the sense we are discussing now; it could be that such an adiabatic process in fact changes the physical identity of the brane.

However, by combining physical and mathematical input, one can still derive Kontsevich's conjecture. Specifically, we need (4.10), the action of the monodromy on the Chern character, and the statement that the monodromy is an autoequivalence. We will appeal to a result of Bridgeland and Maciocia [45] (section 3.3), which states the following. Suppose $\Phi : \mathbf{D}(Y) \to \mathbf{D}(X)$ is a FMT such that for each $y \in Y$, there is a point $f(y) \in X$ with

$$\Phi(\mathscr{O}_y) = \mathscr{O}_{f(y)}\,. \tag{4.14}$$

This induces a morphism $f : Y \to X$ and furthermore the FMT is necessarily of the form

$$\Phi(E) = f_*(\underline{L} \overset{\mathbf{L}}{\otimes} E) \tag{4.15}$$

for any $E \in \mathbf{D}(Y)$, where L is a line bundle on Y, and f_* is the direct image functor.

By composing the true monodromy associated to the conifold point with the inverse of the FMT (4.4), one obtains an autoequivalence which, under our assumptions, is an FMT that preserves the Chern character and every point sheaf $\mathscr{O}_p$. This FMT is covered by the result we just cited with $f(y) = y$. Using the action on the charges to determine $L = \mathscr{O}_Y$, we find that this composition is indeed the identity, establishing the result.

Let us turn to some related comments. It is also instructive to follow $\mathscr{O}$ as we go around the Gepner point five times. This and its generalizations have been analyzed at the Chern character level in papers such as [46]–[51] but not at the derived category level.

Monodromy around the large radius limit is trivially achieved by tensoring with $\mathscr{O}(1)$. One may then consider the combined monodromy transform given by

$$\Psi_G = \left(\mathscr{O}(1) \boxtimes \mathscr{O} \to \underline{\mathscr{O}_\Delta(1)}\right) \tag{4.16}$$

for the combined monodromy around the large radius limit and conifold point. That is, Ψ_G yields monodromy around the Gepner point. One can then compute

$$\begin{aligned}
T_{\Psi_G}(\underline{\mathscr{O}}) &= \left(\mathscr{O}^{\oplus 5} \to \underline{\mathscr{O}(1)}\right)\\
(T_{\Psi_G})^2(\underline{\mathscr{O}}) &= \left(\mathscr{O}^{\oplus 10} \to \mathscr{O}(1)^{\oplus 5} \to \underline{\mathscr{O}(2)}\right)\\
(T_{\Psi_G})^3(\underline{\mathscr{O}}) &= \left(\mathscr{O}^{\oplus 10} \to \mathscr{O}(1)^{\oplus 10} \to \mathscr{O}(2)^{\oplus 5} \to \underline{\mathscr{O}(3)}\right) && (4.17)\\
(T_{\Psi_G})^4(\underline{\mathscr{O}}) &= \left(\mathscr{O}^{\oplus 5} \to \mathscr{O}(1)^{\oplus 10} \to \mathscr{O}(2)^{\oplus 10} \to \mathscr{O}(3)^{\oplus 5} \to \underline{\mathscr{O}(4)}\right). && (4.18)
\end{aligned}$$

The restriction of (3.18) to X can then be used to show that $(T_{\Psi_G})^4(\underline{\mathscr{O}}) = \underline{\mathscr{O}(-1)}[4]$. A final application of T_{Ψ_G} then yields the result

$$(T_{\Psi_G})^5(\underline{\mathscr{O}}) = \underline{\mathscr{O}}[2]\,. \tag{4.19}$$

Indeed it is an unpublished result [52] that

$$(T_{\Psi_G})^5(A) = A[2]\,, \tag{4.20}$$

for any $A \in \mathbf{D}(X)$. In particular "5 times around the Gepner point" is not the identity map on $\mathbf{D}(X)$ but rather a shift. A similar result for the elliptic curve was described in [37].

The analysis of the quintic should extend naturally to any Calabi-Yau manifold represented as a complete intersection in a toric variety. It particular, there is a "primary component of the discriminant locus" in the moduli space which plays the rôle of the conifold point above as discussed by Horja [47] and Morrison [46] and also in [51]. Monodromy around this will then also be given by the object in (4.3).

What about other monodromies for a general Calabi-Yau threefold? The monodromy around the conifold point arises purely because of the simple zero appearing in $Z(\underline{\mathscr{O}})$. It is clear then that the arguments above should generalize easily for any case where a single soliton is becoming massless at a given point in the moduli space. This is characteristic of all "conifold"-like points as argued in [53]. We simply replace $\underline{\mathscr{O}}$ by the particular object $S \in \mathbf{D}(X)$ becoming massless. This yields

$$\Psi = \mathrm{Cone}\left(S^\vee \overset{\mathbf{L}}{\boxtimes} S \to \underline{\mathscr{O}_\Delta}\right), \tag{4.21}$$

as the corresponding object generating the Fourier–Mukai transform. It would be nice to establish the more general monodromies discussed by Horja [54].

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5. An application to supersymmetry breaking

Let us conclude by briefly describing one possible application of these ideas, namely to finding $\mathscr{N} = 1$ compactifications of string theory which spontaneously break supersymmetry.

In the general context of compactification with branes, a popular way to get supersymmetry breaking is to incorporate combinations of BPS space-filling branes preserving different supersymmetries, for example a brane and its antibrane. One could incorporate non-BPS branes for the same purpose.

Let us discuss this type of construction in the context at hand of tree level type-II compactification. As is well known, in this limit BPS brane considerations do not depend on the number of Minkowski dimensions filled by the branes, so we can apply Π-stability as discussed here to this problem, and say that a compactification will break supersymmetry in this way if the K theory class of the embedded D-branes do not support a stable or semi-stable configuration. By semi-stable we mean a direct sum of BPS branes with aligned central charge, so this is just a rephrasing of the problem.

The value of this rephrasing is that, while it is not hard to find combinations of BPS branes which break supersymmetry for a particular value of the Calabi-Yau and brane moduli, it is quite hard to find combinations which break supersymmetry for all values of the moduli. Indeed we are not aware of any example in the literature. Since the Calabi-Yau moduli are of course dynamical fields in the compactified theory, one must remove supersymmetric vacua at all points in moduli space to truly break supersymmetry.

To give the simplest example of what can go wrong, if one includes a brane and its own antibrane, typically after varying moduli the two will annihilate, leaving a supersymmetric configuration. This is typically signaled by a tachyon in the spectrum from the start and is thus not so interesting. Many examples are known in which no tachyon is visible in the spectrum, at least for the particular Calabi-Yau moduli for which the configuration was constructed, and yet supersymmetry is broken. We discussed the familiar example of the D0-D6 system in section 4, and the techniques here provide many generalizations of this.

The D0-D6 configuration however does not pass the test of breaking supersymmetry for all values of the moduli; on the ms-line we exhibited in figure 7, supersymmetry is restored. This phenomenon is very generic: for example, any compactification with two types of BPS brane would be expected to restore supersymmetry on a wall of real codimension 1. This also points to the larger problem, which is that since BPS central charges vary drastically with the moduli, an analysis in the neighborhood of some point in moduli space is not at all adequate to address this problem. A global picture is needed.

We note in passing that a consistent string compactification with D-branes must of course include orientifold planes and cancel RR tadpoles, and that one would be particularly interested in finding a brane configuration of this type which breaks supersymmetry. At present it appears that breaking supersymmetry is the hard part of the problem, while numerous examples of tadpole canceling configurations are known.

Using the ideas discussed in subsection 2.6 and in [27], one can see that in the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold (and probably in general $\mathbb{C}^3/\Gamma$ orbifolds), one can never break supersymmetry in this way. In other words, for every RR charge vector, there is some point in Teichmüller space $\mathscr{T}$ at which the configuration is semi-stable, and supersymmetry is restored.

To analyze this problem, it is simplest to express the RR charges in the basis of fractional branes discussed in [55, 27]. The simplest case to consider is then when all charges in this basis are non-negative, or all are non-positive, as then it is manifest that supersymmetry is restored at the orbifold point: one simply considers the appropriate direct sum of fractional branes, all of which preserve the same supersymmetry. The result holds even for general charges, because by moving on the Teichmüller space one can induce a monodromy after which all charges become positive in this basis. It suffices to consider the monodromy corresponding to $B \to B + n$ for sufficiently large n; by results cited in [27] the resulting charge vector is $n_i = H^0(M, V \otimes O(i - 3 + n)) \geq 0$ for $i = 1, 2, 3$ and is manifestly non-negative. Thus there will exist monodromy images of the "original" orbifold point at which supersymmetry is restored.

The main ingredient which is needed to get this very general result is that there exist a point in moduli space at which a set of branes forming a basis for the entire derived category of branes (such as the fractional branes in the orbifold problem) all have aligned central charges. As we discussed in subsection 2.6, this is probably not true for compact Calabi-Yau, leaving the possibility of supersymmetry breaking by this mechanism open. It would be quite interesting to find a concrete example.

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